

# QCD at Non-Zero Density : Lattice Results

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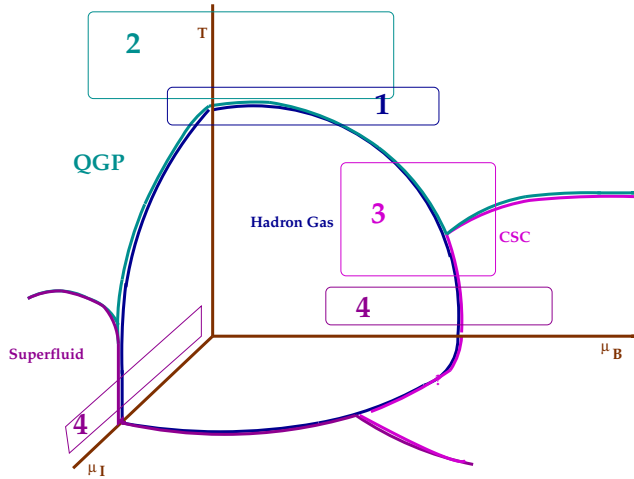
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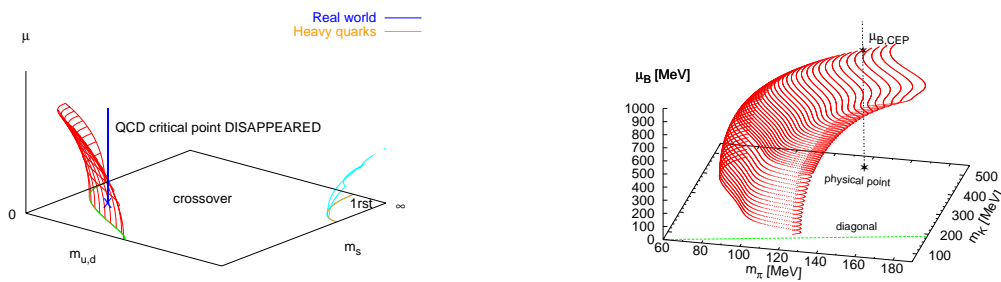
**Abstract.** A concise review of the progress of lattice calculations at non-zero density since QM2006, with emphasis on the high baryon density, low temperature domain. Possibilities for exploring densities higher than those studied by standard techniques are analysed. The phase transitions of cold, dense matter, where the sign problem remains severe, are discussed in the context of QCD-like models and approximations to QCD.

## 1. Introduction

The subject of this note is illustrated in Fig. 1, where I have sketched the phase diagram of QCD in the temperature, baryochemical potential, isospin chemical potential space. Lattice calculations aim at a quantitative analysis of the QCD phase diagram, using the QCD Lagrangian as a sole input. I will review the progress towards this goal since QM2006[1] examining the various thermodynamic regions indicated in Fig.1.



**Figure 1.** The phase diagram of QCD in the  $T, \mu_B, \mu_I$  plane. Mature calculations are possible around  $T_c$ , for small  $\mu_B$ , in the region marked (1) in the plot, and in the domain of the strongly interactive QGP (2). Progress is being made in the parameter range of potential interest to FAIR (3), while the physics of cold, dense matter (4) can be explored only with a nonzero isospin density, or by use of two-color QCD.



**Figure 2.** The critical surface of QCD, if the current results obtained on moderately fine lattices would persist in the continuum limit (left, after [4]); the critical surface of a  $SU(3)XSU(3)$  chiral model, from [7], which contains the endpoint of the chiral line in the  $T, \mu_B$  plane.

## 2. The Critical line and the Critical Point

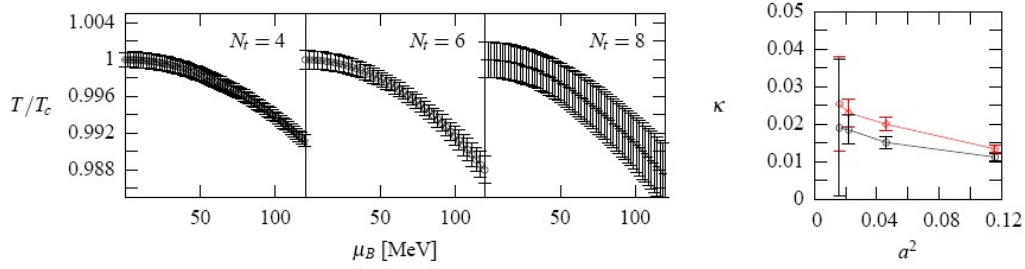
A critical point in the  $T, \mu_B$  plane has been predicted by model calculations, and several lattice studies have searched for it, without finding a general agreement, see e.g. [2] for a review. In ref.[3] an alternative strategy was proposed, based on the analysis of the slope  $K$  of the critical surfaces in the  $m_{u,d}, m_s, \mu_B$  space stemming from the critical line which limits the first order transition area of the  $m_{u,d}, m_s$  plane:

$$m_c(\mu) = 1 + K(\mu/T)^2 \quad (1)$$

A positive slope would be indicative of a critical endpoint.  $K$  was computed for three degenerate quark masses  $m_{u,d} = m_s$ , giving  $K = -3.3(5)$  [4]. An analogous calculation, but for isospin baryochemical potential, give  $K = -3.0(1)$  [5]. Close to the zero density critical temperature the difference between effects produced by baryochemical potential and isospin chemical potential is small ( for instance, large  $N_c$  calculation predict differences  $O(1/N_c^2)$  [6]), so the two results [4, 5] consistently suggest that the region of first order phase transition becomes smaller at nonzero chemical potential, at least till  $\mu < 600$  MeV. A mean field calculation carried out in the  $SU(3)_LXSU(3)_R$  NJL model shows instead the opposite trend[7], see Fig. 2, right. All in all we are observing qualitative differences between QCD and purely fermionic models at small chemical potential. Indeed chiral symmetry is most likely broken by long distance forces in QCD, and by short distance vector forces in NJL, and these differences might well have an impact on the existence of the critical endpoint, see also ref. [8].

The results [4, 5] have been obtained on lattices with  $N_t = 4$ . It would be very important to confirm these predictions closer to the continuum limit. Indeed, the relevance of taking the continuum limit can be hardly overemphasised in QCD [9] .

One substantial step towards the continuum limit of the critical line has been presented at this meeting [10]. The slope of the critical line has been computed for  $N_t = 4, 6, 8, 10$ , and the claim is that there is a reasonable control over the  $a \rightarrow 0$  extrapolation, see Fig.3. Note that while the (pseudo)critical temperatures



**Figure 3.** The approach to the continuum limit of the critical line [10]. The critical lines computed on lattices with size  $N_t = 4, N_t = 6, N_t = 8$  are shown in the left plot, while the slope of the critical line, computed from two different susceptibilities, as a function of the lattice spacing are summarised in the right plot. Courtesy C. Guse.

at  $\mu = 0.0$  estimated from the peak of the Polyakov loop susceptibility differs from the one estimated from the peak of the chiral susceptibility, the slope of the two lines are the same within errorbars.

### 3. Equation of State and Critical Behaviour

Thermodynamics studies on the lattice are based on the analysis of the number density  $n_{u,d}(T, \mu_u, \mu_d, m_u, m_d) = \frac{\partial p(T, \mu_u, \mu_d)}{\partial \mu_{u,d}}$  and its associated susceptibilities

$$\chi_{j_u, j_d}(T) = \frac{\partial^{(j_u + j_d)} p(T, \mu_u, \mu_d)}{\partial \mu_u^{j_u} \partial \mu_d^{j_d}} \Big|_{\mu_u = \mu_d = 0}. \quad (2)$$

The latter are a significant probe of the fluctuational behaviour of the system[11, 12], and are also the Taylor coefficients of the excess pressure  $\Delta p(T, \mu_u, \mu_d) \equiv p(T, \mu_u, \mu_d) - p(T, \mu_u = 0, \mu_d = 0)$  which contains information about baryon density effects in the EoS. By exploiting these observables it was established already at the time of QM2006 that the hadron resonance gas model, where  $n(T, \mu) = K(T) \sinh(N_c \mu/T)$  describes well the system up to rather high temperatures  $T \simeq 0.95T_c$  [14, 15].

For zero chemical potential, new results for the susceptibilities have been presented by the RBC-Bielefeld collaboration[16]. They confirm the hadron gas behaviour at low temperature, and the approach to the free gas at high temperature. Most important, they note that these behaviours are not smoothly connected, rather there is a peak in between associated with the critical behaviour, see Fig. 4.

At nonzero chemical potential, it turns out that it is useful to consider the phase diagram of QCD in the  $T, \mu^2$  plane. It has been found [17] that the strongly interactive quark gluon plasma can be described by  $p(T, \mu) = b(T)|t + a(T)(\mu^2 + \mu_c^2)|^{(2-\alpha)}$ , implying  $n(T, \mu) = A(T)\mu(\mu_c^2 + \mu^2)^{(2-\alpha)}$ , where  $\mu_c$  is the critical point at imaginary chemical potential, see Fig. 5. Note that the simple polynomial behaviour of the free field limit would be recovered when  $\alpha = 1$ , and that  $\alpha > 1$ , resulting from the numerical analysis, implies a slower increase of the particle number in the critical region with respect to the free case. It would be interesting to repeat this investigation using the

generalised method of ref.[18]. Data at imaginary chemical potential are amenable to an easy comparison with analytic studies, and indeed the results in the strongly interactive region compares favourably with a quasiparticle study[19] , once an explicit dependence of the self-energy parts on  $\mu_i = \mu_{u,d}$  and  $T$ , as well as an implicit dependence via the effective coupling  $G^2(T, \mu_u, \mu_d)$  has been taken into account:

$$\omega_i^2 = k^2 + m_i^2 + \Pi_i, \quad \Pi_i = \frac{1}{3} \left( T^2 + \frac{\mu_i^2}{\pi^2} \right) G^2(T, \mu_u, \mu_d). \quad (3)$$

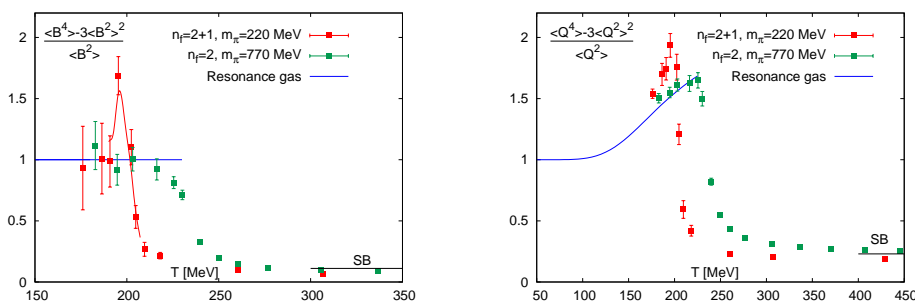
At larger temperature the data should compare successfully with a resummed perturbation theory [20], and it remains to be seen which is the lower limit of its applicability.

#### 4. Towards Fair

While the results of the two previous sections are robust, and we believe they can be systematically improved much in the same way as standard lattice QCD calculations, in this colder, denser region the sign problem becomes more severe and the results should still be considered exploratory [21].

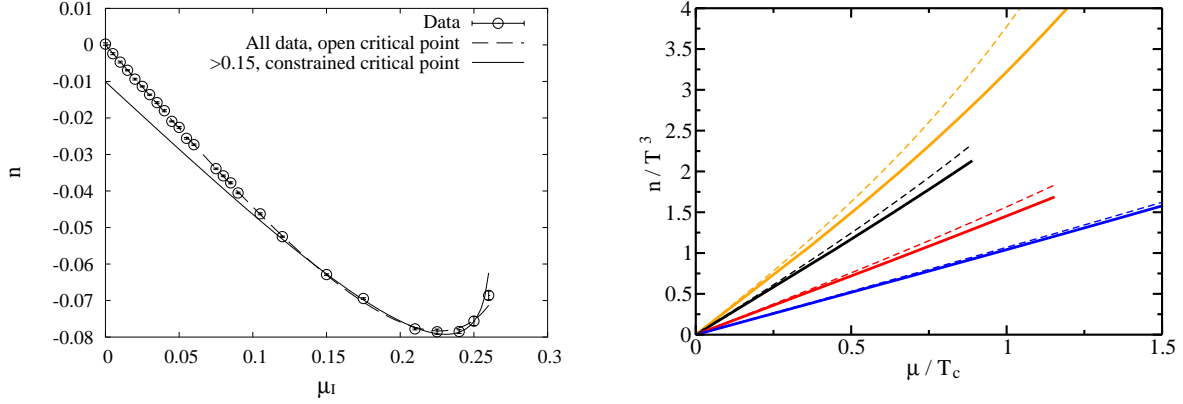
The mass spectrum has been computed via the QCD strong coupling expansion, observing the expected signatures of chiral symmetry restoration, and a  $\rho$  mass decreasing with temperature[22]. Calculations have also been performed in the double limit:  $M \rightarrow \infty, \mu \rightarrow \infty, \zeta \equiv \exp(\mu - \ln M)$  fixed , corresponding to an evolved ‘quenched approximation’ in the presence of charged matter. The order parameter has been computed (note the ridge in the  $T, \mu$  plane in Fig. 6, left diagram), allowing the observation of the phase transition . There are also emerging indication of a tricritical point, and studies of diquark are in progress [23]. Similar results come from a study using overlap fermions at strong coupling [24].

New results in the canonical formalism, in which the canonical partition function

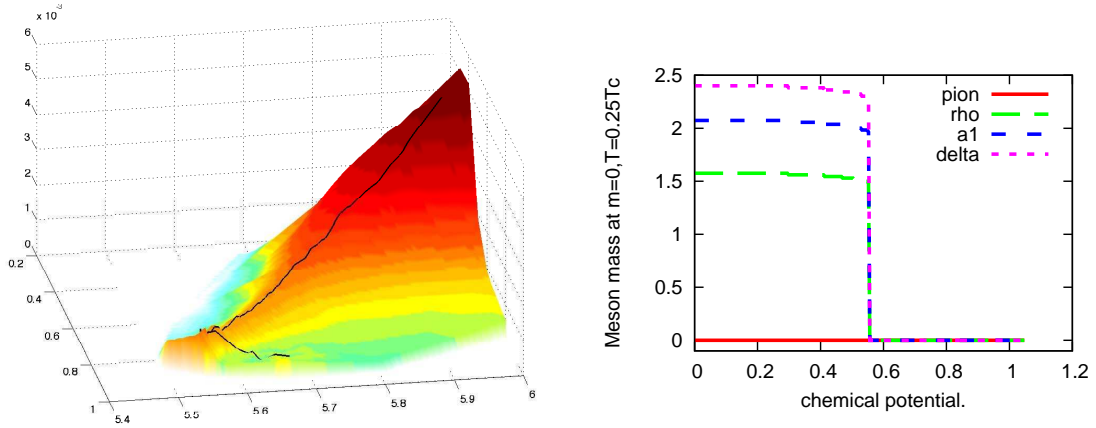


**Figure 4.** Susceptibilities as a function of temperature [16] demonstrating that the critical region of real QCD region cannot be described by simple models. Courtesy C. Schmidt

### QCD at non-zero density



**Figure 5.** The particle density as a function of imaginary chemical potential, at  $T = 1.1T_c$  is well fitted by a conventional critical behaviour (left, from ref. [17]), as well as by a quasiparticle model with  $\mu$  dependent coefficients[19]. Once continued to real chemical potential, this gives a modified form of the Stefan-Boltzmann law (right, upper curve, from [19]), which by increasing  $T$  gradually approaches the free field behaviour, corresponding to the lower curve in the plot



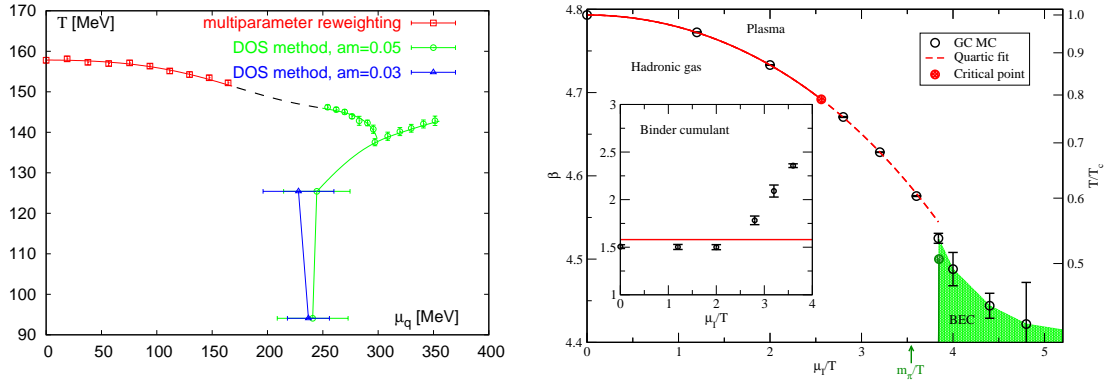
**Figure 6.** Results from QCD expansions: The Polyakov loop as a function of  $T, \mu$  from a numerical study at large mass(left [23]) and the pattern of chiral symmetry as seen from the meson spectrum at low temperature as a function of  $\mu$  from strong coupling QCD (right,[22])

was evaluated following Hasenfratz-Toussaint[25]:

$$\begin{aligned} \frac{Z_C(B, \beta)}{Z_{GC}(\beta_0 = \beta, \mu = i\mu_{I_0})} &= \left\langle \frac{1}{2\pi} \int_{-\pi}^{\pi} d\left(\frac{\mu_I}{T}\right) e^{-i3B\frac{\mu_I}{T}} \frac{\det(U; i\mu_I)}{\det(U; i\mu_{I_0})} \right\rangle_{\beta_0, i\mu_{I_0}} \\ &\equiv \left\langle \frac{\hat{Z}_C(U; B)}{\det(U; i\mu_{I_0})} \right\rangle_{\beta_0, i\mu_{I_0}} \end{aligned} \quad (4)$$

allowed the identification of a first order line, a coexistence region, and the associated critical  $\mu$  and critical densities, for four-flavor QCD[26].

A variant of the canonical approach, based on the evaluation of the Grand Partition Function via a Taylor expansion produced results for  $N_f = 2$ , indicating a qualitative change at  $T/T_c \simeq 0.8$ , and a first order line [27].



**Figure 7.** The phase diagram of QCD in the  $T, \mu_B$  plane, from a (still exploratory) Density of States calculation (left diagram [28]). The phase diagram of QCD in the  $T, \mu_I$  plane (right diagram, [30]).

The density of states method - a reordering of the functional integral based on the constrained partition function (which in condensed matter parlance might be called a mesoscopic approach )-

$$\rho(x) = \int \mathcal{D}U g(U) \delta(\phi - x). \quad (5)$$

applied to the  $N_f = 4$  theory allowed the identification of two phase transition lines, and gave indication of a triple point [28] (Fig.7, left).

## 5. Cold and dense matter : QCD-like models

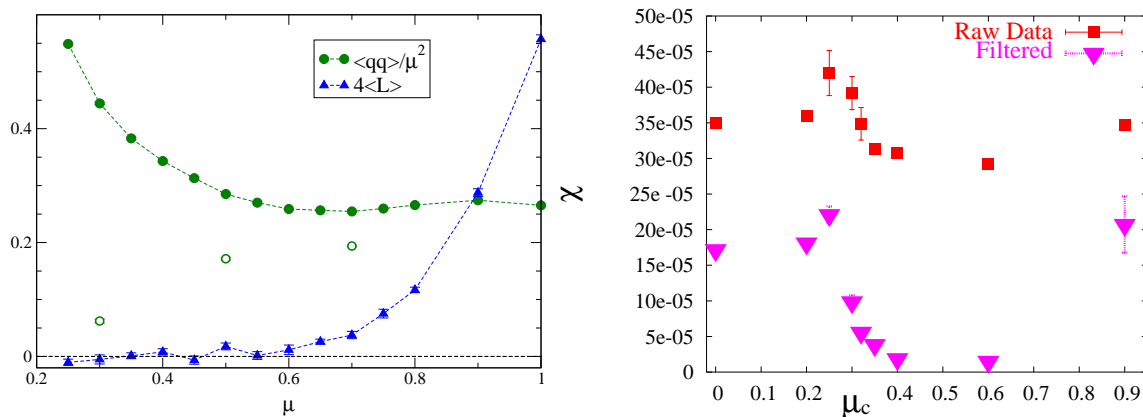
Finally in the cold and dense phase, the sign problem is very severe and at this moment I can only show result for QCD at non-zero density of isospin, and two-color QCD at nonzero baryon density. These theories have extra symmetries, which protect the determinant from becoming complex, and allow standard MonteCarlo simulation, see e.g ref. [29] for reviews.

At zero temperature the onset for thermodynamics equals, as usual, the mass of the lowest excitation carrying the relevant charge, i.e.  $\mu_c = m_\pi/2$  in the case of isospin density, and  $\mu_c = m_B/N_c = m_\pi/2$ , for two- color QCD, so the physics of a finite density of isospin, and of finite baryon density have many analogies.

The phase diagram of QCD in the  $\mu_I, T$  plane has been recently calculated, and the superfluid BEC phase has been clearly identified [30] (Fig.7, right).

Early studies, both analytic and numeric, have identified a superfluid phase also in two-colour QCD, by inspecting the behaviour of fermionic observables, and confirming the prediction of chiral perturbation theory, which is applicable here [29].

There is an emerging consensus that the BEC phase of two-colour QCD is still confining, and preliminary indications of a BEC/BCS crossover have been reported [31],



**Figure 8.** The superfluid phase of two colour QCD : diquarks condense (open symbols are for  $\langle qq \rangle$  extrapolated to the chiral limit), however the Polyakov loop remains close to zero, hence the superfluid phase still confines (left, from ref [31]; the amplitude of the glueball propagator peaks at the critical point, and reaches a lower value above  $\mu_c$ , indicating non trivial modification of the baryon-dense gluonic medium (right, from ref. [33]).

with the superfluid order parameter assuming the scaling consistent with Cooper pairing at a Fermi surface. Results from ongoing work[32] indicate that a BEC/BCS transition persists at a finer lattice spacing. At the hadronic/BEC interface there are signal of criticality in the gluonic sector, as well as non trivial modifications of the glueball propagators in the superfluid phase [33] (Fig.8).

One final comment concerns the large  $N_c$  limit, where a scenario has been proposed in which deconfinement and chiral transition are separated[34]. It is very tempting to speculate on possible analogies of these observation with the behaviour of  $N_c = 2$ , where indeed chiral transition (even if between two phases where chiral symmetry remains broken) and deconfinement transition are different phenomena.

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